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Interview Jan-Willem van Ittersum

Feeling like an explorer

On the 16th of October, mathematician Jan-Willem van Ittersum, who completed his PhD in 2021 at Utrecht University, was awarded the Christiaan Huygens Science Prize for his dissertation “Partitions and quasi-modular forms; variations on the Bloch-Okounkov theorem”. He received the prize - consisting of an amount of ten thousand euros and a bronze sculpture of Christiaan Huygens - from Minister Bruins of Education, Culture and Science. This happened during a festive meeting in Voorburg. Mathematicians Rosa Schwartz and Lucas Slot were also nominated for the award and received an honorary distinction.

To celebrate the occasion, Nicos Starreveld interviewed Jan-Willem to discuss his research, his future ambitions, and the wonderful mathematics problems which keep him motivated and enthusiastic. Currently, Jan-Willem is doing a postdoc at the University of Cologne. Let us travel back in time and follow him in his mathematical steps that led him to his current position.

Jan-Willem, first, my congratulations on receiving this prestigious award. You received your PhD in 2021 on a topic with a very long history, namely partitions of integers. You have found a very special connection between partitions and modular forms, building upon the work of Bloch and Okounkov. Before we discuss the mathematics, I am curious to hear more about how you discovered mathematics. Why did you want to become a researcher?

“Thank you for the nice words, I really enjoyed the evening in Voorburg. The ceremony was very well organized and the atmosphere was great. From a young age I was fascinated by mathematics. I didn’t know what I wanted to do when I would grow up, but I knew I wanted to do something with mathematics. This fascination grew stronger during my studies at Utrecht University. The first time I came in contact with research was during my bachelor thesis, in algebraic number theory.

My supervisor, Gunther Cornelissen, asked me to read an article because there was a sentence he did not understand. I found the article very interesting; I started experimenting and working out examples to understand the results better. At some point I also thought I had proven a theorem, but it turned out I misunderstood some concepts. But this is how it goes with research. At the end, I managed to prove a new result, which was very nice. This was my first encounter with research, and I knew that I wanted to keep doing it and keep exploring open questions.”

Gunther was also your supervisor during your master thesis, and later also your promotor, together with Professor Don Zagier from the Max Planck Institute for Mathematics in Bonn. Could you say something about your collaboration?

“After my bachelor I already knew I wanted to continue as a researcher. When I was finishing my master Utrecht University gave the opportunity to students to submit their own research proposal for a PhD project, which was possible because the university had received a grant from NWO. This was a great opportunity.

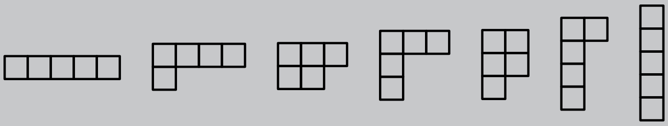
Advised by Gunther and Don, I found a very interesting problem to work on and I submitted a proposal, which was accepted.

The collaboration with Gunther was always very pleasant and motivating. Something I find unique in him is how he collaborates with students. He is very good in tailoring a project to the needs and level of a student. He recommended it would be useful to ask Don as the second supervisor for my research, since the topics I was interested in were related to research Don had done in the past.

The topic of my PhD was inspired by an article Don had written. The article was motivated by some problems from geometry. Don he made a translation to a problem about partitions, which he then related to modular forms. I found this relation of partitions and modular forms very interesting, and we decided to investigate it further.”

Understanding partitions

Central to Jan-Willem's research is the concept of a partition of a number. A partition of an integer is a way to express the number as a sum of positive integers, where the order of the terms does not matter. For example, the partitions of 5 are given by 5, 4+1, 3+2, 3+1+1, 2+2+1, 2+1+1+1, and 1+1+1+1+1, which can be represented graphically by their Young diagrams as



Partitions of numbers have intrigued mathematicians since the 18th century. For instance, is there a systematic way to find all partitions of a natural number? Such questions quickly proved to be very challenging, even for today's most powerful computers. To get an idea: the number 100 has 190,569,292 partitions. If you try to find them all, then you are probably doomed to make a mistake or get exhausted and never want to do math again!

Mathematicians therefore had to be creative to better understand partitions. Leonard Euler was the first to use functions to study partitions. He developped a method to determine the number of partitions of a number by using power series. Euler proved that

$$\sum_{n=0}^{\infty} p(n) q^n = \prod_{j=1}^{\infty} \frac{1}{1-q^j}. \qquad (1)$$

To prove this formula, and to show why it is useful in understanding partitions, Euler had to work on it for many years. By working on the infinite product on the right he managed to prove a wonderful result, the Euler pentagonal theorem. This theorem gives a recursive expression to compute partitions, namely

$$p(n) = p(n-1) + p(n-2) - p(n-5) - p(n-7) - \cdots = \sum_{k \neq 0} (-1)^{k-1} p(n - g_k).$$

where the summation is over all nonzero integers k (positive and negative) and g_k is the k-th generalized pentagonal number. The infinite sum is in practice a finite sum, because $p(n) < 0$ for negative numbers. For many years this recursive formula was the only way to compute and work with partitions. Until 1918, when Hardy and Ramanujan proved the asymptotic expression

$$p(n) \sim \frac{1}{4n\sqrt{n}} \exp\left(\pi \sqrt{\frac{2n}{3}}\right).$$

You can experiment yourself with this formula and investigate how good it approximates $p(n)$. In Wolfram Alpha you can compute the approximation of the summation for rather high values of n. Ramanujan proved many results about partitions. A collection of his work relying on modular arithmetic bears the name "Ramanujan's congruences". Furthermore, in 1937 Hans Rademacher improved this asymptotic result of Hardy and Ramanujan by providing a convergent series expression for $p(n)$, which is considered up to today to be the best closed form expression possible.

Foto: Paul Voorham

Jan-Willem van Ittersum received the Christiaan Huygens Science Prize. Rosa Schwartz and Lucas Slot received an honorary distinction.

What is the motivation to study this bracket (see equation (2) on the next page)?

"From a mathematical perspective, you should interpret the q-bracket in the following sense: suppose you want to investigate a sequence of numbers. On most occasions, the best way to start is by using the generating series of the sequence, where the terms of the sequence are the coefficients of each term, like Euler did in computing the generating series of partitions, for example. Then you use the obtained results to deduce information about the sequence. What is very surprising and interesting is that only for very specific choices of weight-functions the q-bracket is a (quasi)modular form.

During my research I wanted to understand for which families of weight-functions the q-bracket will constitute a family of quasimodular forms. Bloch and Okounkov had proven this for shifted symmetric functions, I proved it also for other cases, for example symmetric functions. If there are more such families is still an open question.

Here in Cologne we are also interested in the modular properties of the q-bracket; for instance, we can use so-called mock-modular forms to prove some new results. Cologne is in some sense a center for research in modular forms, with two professors who are experts in this topic, Kathrin Bringmann and Sander Zwegers."

Did you feel well prepared to work on this topic when you started on your PhD?

"When I started, I found it very exciting and a little bit scary. But I had followed a course on Modular Forms by Sander Dahmen and Peter Bruin during my master, which triggered my interest on this topic. It also helped a lot that I could apply for the position by submitting my own proposal. The process of working on this topic was very exciting and I enjoyed it a lot. From the first day, I tried to work out many examples to understand how it works. And understanding good examples turned out to be pivotal for my research. In the literature we could find various specific examples of functions which lead to quasimodular forms. But, besides the result of Bloch and Okounkov, it was not known whether you could find an algebra of functions which leads to an algebra of quasimodular forms. To build confidence and get insights on how this could work we had to work out many concrete examples. I remember that when I was still at school, one specific mathematics teacher stimulated me to participate in the first round of the Mathematical Olympiad. I performed well and I got invited to a series of trainings. There I learned how to approach mathematical problems where I had no idea how to start. These problem-solving skills are very important, also today in my research."

Which research direction did you choose after your PhD?

"Towards the end of my PhD, I was discussing with mathematicians at conferences about interesting topics to work on. One of these mathematicians was Henrik Bachmann, who got me interested in the so-called MacMahon's q-series, which are also related to partitions and modular forms.

In another conference I was discussing with Ken Ono about quasimodular forms and their properties, when we realized there was a very interesting connection between MacMahon's series and prime numbers. We got enthusiastic about it and decided to work on it further. This collaboration led to the recent publication "Integer partitions detect the primes", where we prove that certain equations involving MacMahon's partition functions can "detect" the primes. In this article we found infinitely many equations, which are linear combinations of MacMahon's parti-

Partitions in the 21st century

Back to today. Since the time of Euler and later Ramanujan mathematicians have been working on partitions and have developed new methods to understand them. From the results presented above, it is evident that heavy machinery is necessary to extract information about them. But there is a cost: the more you want to understand, the more advanced the methods become. There is no royal road to understanding them. The information about partitions is hidden within various other objects. Moreover, the kind of information you can extract about partitions also varies. Euler proved a recursive formula, while Ramanujan and Hardy obtained an asymptotic result. But we can also investigate deeper and more subtle properties of $\mathrm{p(n)}$, instead of its value. Mathematicians have found patterns and connections with other well understood mathematical concepts which can yield information about partitions.

The theorem of Bloch-Okounkov is such an example: it provides a connection between partitions and modular forms. This connection between partitions and modular forms was observed before. Namely, the product Euler found in (1) is, up to a factor of $\mathrm{q}^{1/24}$, equal to the modular Dedekind eta-function. This relation was important in the work of Hardy and Ramanujan to prove the asymptotic expression for $\mathrm{p(n)}$. But the Bloch-Okounkov theorem pushes this framework further.
A modern technique to study partitions, which is also central in Jan-Willem's research relies on the q-bracket, defined as

$$< f >_q = \frac{\sum_{\lambda \in p} f(\lambda) q^{|\lambda|}}{\sum_{\lambda \in p} q^{|\lambda|}}. \qquad (2)$$

where $\mathscr{P}$ is the collection of all partitions. You can view the sum in the numerator as the infinite sum

$$\sum_{n=1}^{\infty} \sum_{\lambda \vdash n} f(\lambda) q^{|\lambda|}.$$

We are interested in the functions f for which $< f >_q$ is a nice function with good algebraic properties. We can notice that the denominator in the definition of $< f >_q$ is exactly the power series in (1), which is known by the work of Euler. In his dissertation Jan-Willem proved that for the family of symmetric functions the q-bracket is a quasimodular form. Which means that it inherits many nice properties and symmetries that these well understood mathematical objects have.
For a more mathematical and detailed overview of Jan-Willem's work we refer the reader to the nice overview he wrote for the Nieuw Archief voor Wiskunde in 2022 [1].

tion functions, which are zero if and only if you plug in a prime number. This was a surprising result, because there is no evident reason why partitions should be related to the prime numbers! Partitions are related to addition, while prime numbers are related to multiplication and factorization. It is very interesting that there is a connection between them."

From your experiences until now, which are the important open questions in your field at the moment?

"Modular forms are very well understood. Most open problems about modular forms concern their relations to other mathematical objects, for example in the Langlands program. An open question which I am very interested in concerns families of functions which are not exactly modular, but look like such forms. There are nowadays various types of modularity, for example quasi-, mock or quantum modular forms. I am quite interested in such variations of modularity, in particular for series for which the precise modularity is not yet understood."

Is mathematics something you do on your own or working together in a team?

"I think you need to find a balance, between understanding the topic on your own and collaborating or discussing with colleagues. Being in a team also helps when you get stuck, which naturally occurs in research. Often it helps a lot to push yourself to gather your ideas, organise them, and explain them to someone else. When you work independently you get stuck in your own ideas sometimes. Discussing with other mathematicians helps gain new insights and ideas. If I reflect on my previous projects, the projects where I collaborated with other mathematicians are those I enjoyed the most.

In general, I often feel like an explorer when I am investigating a new mathematical problem, similar to the analogy Andrew Wiles shared in the BBC Documentary about Fermat's Last Theorem [2]. He mentioned that when you start with a new problem, you feel like you are in a dark room and you don't see anything in it, but as you start exploring you start touching objects and it becomes clearer how the room looks like. Doing mathematics feels like this, like an explorer discovering new things which sometimes reveal a great beauty. Whether I do it on my own or with colleagues, this is always the part I enjoy the most, the feeling of being an explorer!"

References

1 Een recept voor quasimodulaire vormen, Jan Willem van Ittersum, *Nieuw Archief voor Wiskunde*, September 2022.

2 BBC, *Horizon*, 1995-1996, Fermat's Last Theorem.

3 Math Is Still Catching Up to the Mysterious Genius of Srinivasa Ramanujan, Jordana Cepelewicz, *Quanta Magazine*, October 2024.