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Inaugural Lecture

The mathematics of

On 3 May 2023 Antonis Papapantoleon delivered his inaugural lecture at TU Delft. This text is based on this lecture. It offers a gentle introduction to mathematical finance, with emphasis on the valuation of derivatives and the modeling of asset prices.

A motivating example: Games with coins

Assume that I have a €1 coin in my toga and I am offering you the possibility of playing one of the following two games:

- Game A: in case Head appears I will give you one Euro, while in case Tail appears you will give me one Euro.
- Game B: in case Head appears I will give you again one Euro, but in case Tail appears nothing will happen.

A natural question to ask first, is *which game would you prefer to play*, if you were given the choice between Game A and Game B. The answer is rather obvious: any rational individual would prefer to play Game B, since she will be given the opportunity to gain one Euro in case Head appears, but she will not lose anything in case Tail appears.

Figure 1 Dutch Euro coins.

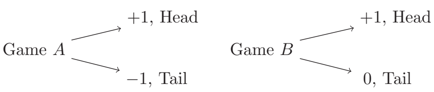

Figure 2 Payoffs of Game A and Game B.

However, as I am standing on the other side of this game, I only stand to lose by playing Game B. Hence, in order to start playing this game I will ask you for an 'entrance fee'. Let us now compute the *fair price for entering Game B*, where by *fair* we mean the price at which both sides will be happy and will start playing the game.

The data describing the outcome of this game are the following:

- Head or Tail appear with probability $\frac{1}{2}$, assuming that the coin is *fair*.
- Head: you gain €1 and I lose €1.
- Tail: no gain or loss for both sides.

Then, we can agree that the fair price for entering the game equals the expected gain (for you) or the expected loss (for me) based on the data above, which equals

$$\begin{aligned}\text{Price} &= \text{Expected Gain}\\ &= \frac{1}{2}\times(+1)+\frac{1}{2}\times 0 = \frac{1}{2}.\end{aligned}$$

In mathematical terms, we can summarize this computation as follows: the fair price for entering the game equals the expected gain of the game (B) under a probability measure ($\mathbb{Q}$) that makes the game fair, *i.e.*

$$\text{Price} = \mathbb{E}_{\mathbb{Q}}[B]. \tag{1}$$

This toy example already embodies the main tasks of a mathematician working in financial markets:

- Modeling, of a game or a random outcome in general.
- Estimation, of the parameters governing this game or random outcome.
- Computation, of the fair price of the game or expected gain of the random outcome.

However, before we start working on these tasks we need some *theory*.

The origins of financial mathematics

Louis Bachelier (1870–1946) was born in a family of wine merchants and bankers, which played a particular role in his later career. Because he lost his family at a young age, he worked at the family busi-

Figure 3 Louis Bachelier.

Antonis Papapantoleon

financial markets

ness, and learned about the workings of financial markets from a young age. He worked for his PhD in mathematics under the supervision of Henri Poincaré and, due to his background, he decided to work in the mathematics of financial markets. His PhD thesis was defended in 1900 and was titled *Théorie de la spéculation* (i.e., ‘Theory of Speculation’) [1].

A key idea from Louis Bachelier’s thesis, which summarizes his theory for the valuation of financial contracts, is the following:

> “L’espérance mathématique du spéculateur est nulle”

i.e., the mathematical expectation of a speculator is zero. This statement is impressive in itself, because Bachelier is thinking about the ‘mathematical expectation’ of a random quantity in 1900, while the mathematical foundations of probability theory were laid out by Andrei Kolmogorov thirty years later!

Moreover, if we translate this statement in the setting of the games presented before, then what Bachelier means is that the difference between the expected gain and the price for entering the game should equal zero, which brings us back to equation (1). In other words, Louis Bachelier developed the first version of no-arbitrage pricing, already in 1900!

In order to honor his contributions to the field, the society founded in 1996 by researchers in the newly emerging field of mathematical finance was named the *Bachelier Finance Society*. More information about his life and work is available here: https://www.bachelierfinance.org/louis-bachelier.

The basics of financial derivatives

Let us now take a step back and discuss about financial markets and financial derivatives. The origins of derivatives, i.e., contracts whose price depends on the development of the price of another good, can already be traced back to the trading of commodities in ancient Mesopotatmia and medieval Europe; see, for example, the infamous ‘tulipmania’ in Holland during the 1600s.

Modern financial derivatives are legal contracts between counter-parties, that specify certain cash-flows based on the evolution of the prices of some underlying financial assets. Revisiting the games discussed in the beginning, Game A or B is the ‘derivative’, the ‘underlying asset’ is the coin, and the ‘cash-flow’ in, e.g., Game B is the payoff of €1 or €0, depending on the outcome of the coin toss.

Let me now present some basic examples of financial derivatives. AFC Ajax, is a famous football club, but it is also a listed company at the Euronext Amsterdam stock exchange. Therefore, there exist derivatives written on the AFC Ajax stock. The price of the Ajax stock on the day of this address was (approximately) equal to €10.95.

Example 1. The simplest example of a financial derivative is a *forward* contract, which is the obligation to buy 100 shares of the Ajax stock for €$10{,}95$ in three months from now. Assuming that in three months the price of the Ajax stock has dropped to €9.95, then the buyer of the forward contract will make a loss of €100 from this contract. Therefore, a forward contract is analogous to Game A, since they payoff might be positive or negative; see Figure 4.

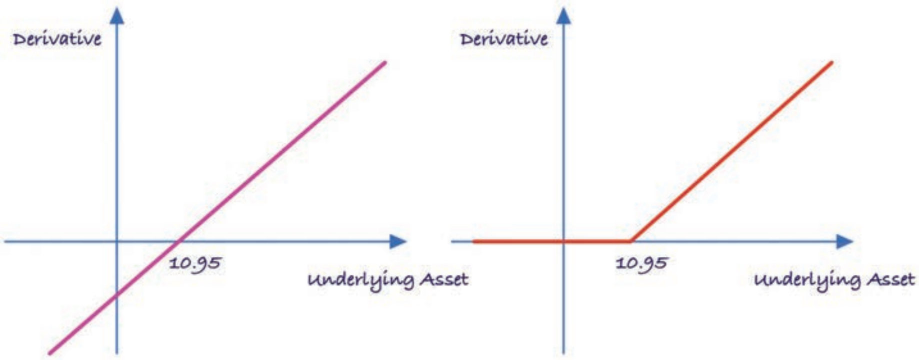

Figure 4 Payoff of a forward contract (left) and an option (right).

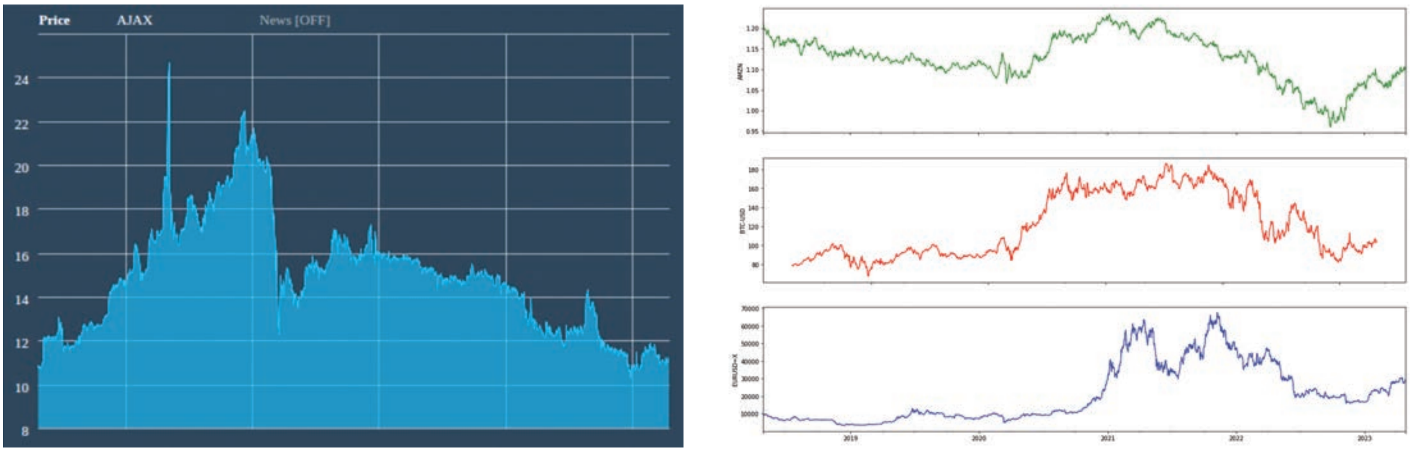

Figure 5 Evolution of the Ajax stock (left), and the Amazon stock, bitcoin and EUR/USD exchange rate from 2018 until 2023 (right).

Example 2. The second simplest example of a financial derivative is an *option*, which is the right, but *not* the obligation, to buy 100 shares of the Ajax stock for €$10{,}95$ in three months from now. Assuming again that in three months the price of the Ajax stock has dropped to €9.95, then the buyer of the option contract will simply not exercise her right, she will buy the stock from the exchange for the market price and will not make a loss from this contract. Therefore, an option is analogous to Game B, since the payoff is always non-negative; see Figure 4.

In mathematical terms, the payoff of an option written on an underlying asset S (e.g., the Ajax stock), with maturity T (e.g., three months from now) and strike price K (e.g., €9.95) equals

$$(S_T - K)^+.$$

An option is guaranteeing its buyer that she will not make a loss, hence the seller of the option will require an 'entrance fee' in order to sell the option, similar to Game B. According to Bachelier's theory, the price of the option should equal

$$\text{Price} = \mathbb{E}_{\mathbb{Q}}[(S_T - K)^+]. \tag{2}$$

We need thus to model the evolution of the Ajax stock price, before computing this price.

Modeling asset prices

We are interested now in modeling the evolution of the prices of financial assets using some mathematical objects known as stochastic processes. Figure 5 presents the evolution of the Ajax stock, the Amazon stock, the price of the Bitcoin cryptocurrency, and the price of the EUR / USD exchange rate from 2018 until 2023.

Although these plots correspond to the prices of financial assets with different characteristics, for example, Amazon is a global company and the bitcoin is a cryptocurrency, the evolution of their prices presents (visually) some common characteristics; namely it is random and rough.

Let us now try to built a model for the evolution of asset prices that has similar characteristics. We can start by tossing a coin, and assigning a higher value in case Head appears and a lower value in case Tail appears. Obviously doing this once does not produce a satisfactory model, but if we toss the coin several times and follow the rule of assigning higher values in case Head appears and lower values in case Tail appears, then the space of possible outcomes of this experiment is depicted on the left part of Figure 6. The right part of Figure 6 presents a realization of this experiment, i.e., the outcome of a specific sequence of coin tosses. In the language of stochastic processes, this is called *path of a random walk*.

Now, mathematicians like to think about limits, so we would also like to know what happens in the limit, when we make infinitely many coin tosses and scale the size of up and down movements accordingly. One can prove that, in the limit, the outcome is a stochastic process called *Brownian motion*, and a path of Brownian motion appears in Figure 7. Visually, this path is very similar to the evolution of asset prices; see Figure 5 again.

Pricing financial derivatives

Let $W = (W_t)_{t\geq 0}$ denote a Brownian motion, where W_t denotes the value of this process at time point t. The second key idea in Bachelier's thesis was to use Brownian motion in order to model asset prices. In particular, he proposed the following model for the evolution of an asset price denoted by $S = (S_t)_{t\geq 0}$

$$S_t = S_0 + \mu t + \sigma W_t, \quad t \geq 0, \tag{3}$$

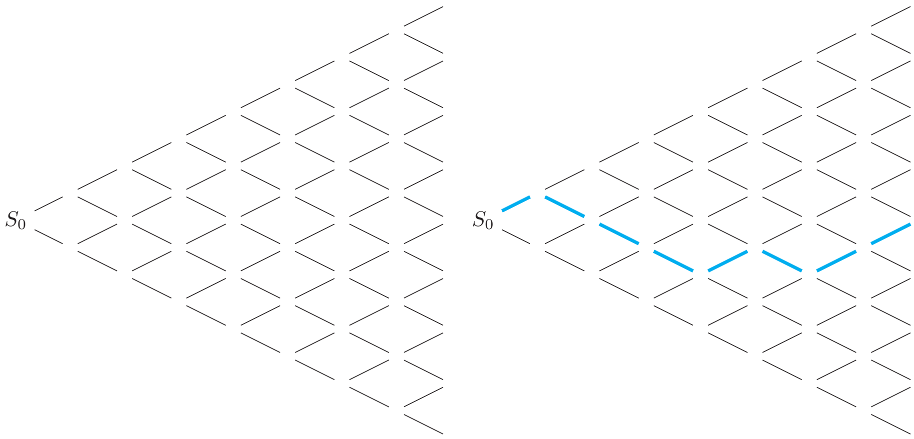

Figure 6 Space of possible outcomes (left) and path of a random walk (right). S_0 denotes the observed price of the financial asset today.

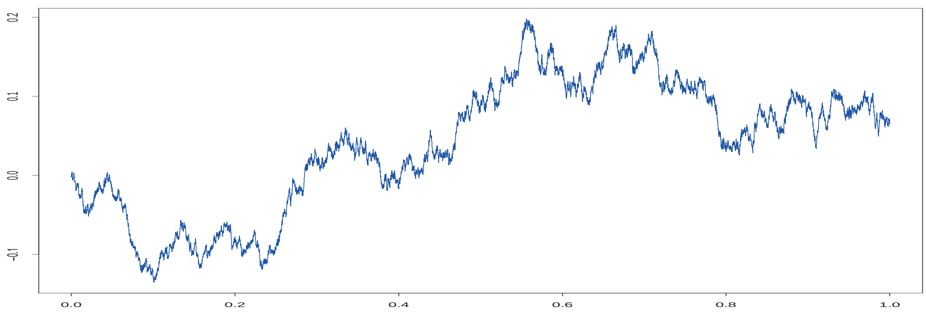

Figure 7 Path of a Brownian motion.

where μ represents some constant trend, σ a positive constant that amplifies the effects of the Brownian motion, while the Brownian motion makes the motion random and rough.

This model was later revisited by the American economist Paul Samuelson in [14], who used the exponential function in order to make sure that asset prices remain positive at all times. The *Samuelson* model for the evolution of asset prices reads then as follows:

$$S_t = S_0 \cdot \exp(\mu t + \sigma W_t), \quad t \geq 0. \qquad (4)$$

The defining moment that led to the widespread use of derivatives in the financial industry came in 1973, when Fischer Black and Myron Scholes in [2] and Robert Merton in [13], published their papers that contained an explicit formula for the price of an option (2) in the Samuelson model (4). More specifically, they showed that the price of an option takes the following form:

Option price:

$$\mathbb{E}_{\mathbb{Q}}(S_T - K)^+ = S_0 \Phi(d_1) - K\Phi(d_2),$$

$$d_{1,2} = \frac{\log\left(\frac{S_0}{K}\right) \pm \frac{\sigma^2}{2}T}{\sigma\sqrt{T}}, \qquad (5)$$

where Φ denotes the cumulative distribution function of the standard normal distribution. This formula relates in an explicit manner on the one hand the variables of the option contract, *i.e.* the strike price K and the time of maturity T, and on the other hand the variables of the model for the evolution of asset prices, *i.e.* the initial stock price S_0 and the volatility σ. Moreover, the appearance of the cdf Φ of the standard normal distribution is related to the properties of Brownian motion, that has normally distributed increments.

This formula has become the market standard for the valuation of financial derivatives, and actually the model in (4) is erroneously referred to as the 'Black–Scholes model', due to the popularity it received following the publication of the formula (5). Myron Scholes and Robert Merton where honored with *The Sveriges Riksbank Prize in Economic Sciences in Memory of Alfred Nobel 1997* for their "new method to determine the value of derivatives". Fisher Black had, unfortunately, passed away the year before, and the prize is not awarded posthumously.

Let us finally mention that, in spite of the popularity of this model, the inability of the normal distribution to describe the empirical behavior of asset prices has been the driving force for intensive academic research. Advanced stochastic models, such as local and stochastic volatility models (e.g., [5, 11]), Lévy processes (e.g., [3, 4, 6, 15]) or rough volatility models (e.g., [7]), that provide a realistic description of asset prices, have attracted the interest of many mathematicians and have led to important advances in the theory of stochastic analysis. Simultaneously, they have been in heavy demand by the financial industry.

Final remarks

The reformulation of the theory of option pricing developed by Black, Scholes and Merton using stochastic integration and martingale theory by Michael Harrison, David Kreps and Stanley Pliska in [8, 9] have turned mathematical finance into an independent scientific field within applied mathematics, with strong interactions with stochastic analysis, statistics, optimization and numerical analysis. Methods developed in all of these fields were utilized to solve problems arising in mathematical finance, while mathematical finance has fed them in return with original, interesting and challenging questions that have led to new research directions. Primary examples of advances motivated by questions in mathematical finance, that showcase its inter- and intra-disciplinary nature, are backward stochastic differential equations (BSDEs) in stochastic analysis (see, e.g., [12]), path-dependent partial differential equations (PDEs) in PDE theory (see, e.g., [16]), and the development of a martingale version of the classical optimal transport problem (see, e.g., [10]).

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