Jan Aarts
Delft University
P.O. Box 5031
2600 GA Delft
johannesaarts@gmail.com

Book review Norman Wildberger: Divine Proportions

From a rational angle

By clever changes of both the distance function and the angle measure a new geometry comes into being. Though the new distance function and angle measure are not linear, in many situations the new trigonometry results is simpler computations. The new theory has unexpected applications in new areas, notably finite geometries. Many classical results of the Euclidean Geometry (that in the good old days were taught in secondary school) have their counterpart in finite geometries. Norman Wildberger presents an outline of the new trigonometry. Jan Aarts discusses the merits of Wildberger's approach.

According to the preface, "This book introduces a remarkable new approach to trigonometry and Euclidean geometry, with dramatic implications for mathematics teaching, industrial applications and the direction of mathematical research in geometry."

The proposed approach to geometry allows for the development of a geometry over any field of characteristic $\neq 2$ very similar to the usual Euclidean geometry. The book is not written as a textbook of rational trigonometry, it is rather intended as an introduction for a mathematically mature audience.

A bird's-eye view of rational geometry

A large part of the book concerns plane geometry. The geometry is defined over a field F with characteristic $\neq 2$ whose elements are called *numbers*. Most of the examples relate to geometries over the field $\mathbf{F}_p$ of the integers modulo the prime number p, the field of the rational numbers, the reals or the complex numbers. A *point* of the plane (over F) is an ordered pair of numbers: $A = [x, y]$ and a *line* is a 3-proportion: $l = \langle a : b : c\rangle$. The point A is on the line l precisely when $ax + by + c = 0$. For example, in the field F_5 of integers modulo 5, the line $l = \langle -2 : 1 : -3\rangle$ passes through the points $[0, 3]$, $[1, 0]$, $[2, 2]$, $[3, 4]$, $[4, 1]$, and no other points.

The basic notions are *parallel* and *perpendicular*, whose definitions do not come as a surprise. The lines $l_1 = \langle a_1 : b_1 : c_1\rangle$ and $l_2 = \langle a_2 : b_2 : c_2\rangle$ are *parallel* if and only if $a_1b_2 - a_2b_1 = 0$. The lines l_1 and l_2 are *perpendicular* if and only if $a_1a_2 + b_1b_2 = 0$.

The crux of the book is that the notions of distance and angle are replaced by quadrance and spread, respectively. The *quadrance* $Q(A, B)$ between the points $A = (x_1, y_1)$ and $B = (x_2, y_2)$ is the square of the 'usual distance', that is

$$Q(A, B) = (x_2 - x_1)^2 + (y_2 - y_2)^2.$$

As said before this theory also applies to finite geometries. However, there is a caveat. In plane geometries over finite fields (or over the complex numbers) there are so-called *null lines* that at first glance have an unusual behavior. The line $l = \langle a : b : c\rangle$ is called a null line if $a^2 + b^2 = 0$. So null lines exist whenever -1 is a square. A null line is parallel to itself (as it should be), but it is also perpendicular to itself. If the points A and B are distinct, then it may be verified that the line AB is a null line if and only if $Q(A, B) = 0$. In the example above, the line l is a null line and the five points on the line l have mutual quadrance 0. If neither of the lines $l_1 = \langle a_1 : b_1 : c_1\rangle$ and $l_2 = \langle a_2 : b_2 : c_2\rangle$ is a null line, the *spread* $s(l_1, l_2)$ between them is the number

$$s(l_1, l_2) = \frac{(a_1b_2 - a_2b_1)^2}{(a_1^2 + b_1^2)(a_2^2 + b_2^2)}.$$

Two lines are parallel if and only if the spread between them is 0. Two lines are perpendicular if and only if the spread between them is 1. If one of the lines is a null line the spread is not defined. The *cross* of lines l_1 and l_2 is defined to be $c = 1 - s$. Later on in the book, it is shown that s and c are equal to the $\sin^2(\alpha)$ and $\cos^2(\alpha)$, respectively, where α is one of the angles enclosed by l_1 and l_2. Note that the spread does not distinguish between supplementary angles.

The most important objects of study are the triangle, its special lines and the circles related to it. If A_1, A_2 and A_3 are three points in the plane (over a field F), we denote by Q_1 the quadrance between A_2 and A_3 and by l_1 the line through the points A_2 and A_3, and similarly for Q_2, l_2 and Q_3, l_3. The triangle $A_1A_2A_3$ is said to be non-null if none of its sides is a null line. The Pythagoras' Theorem holds in the new geometry. It reads as follows: the lines l_1 and l_2 of a non-null triangle

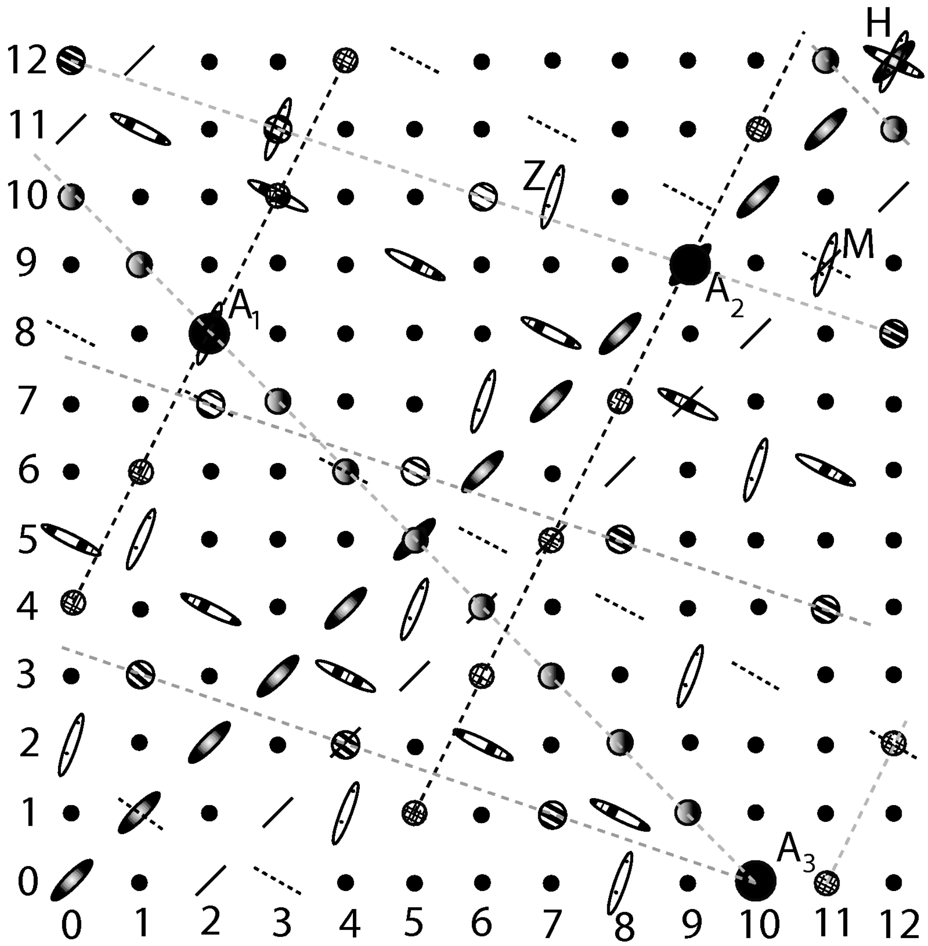


Figure 1 Triangle $A_1A_2A_3$ and some of its special lines and points

are perpendicular if and only if $Q_3 = Q_1 + Q_2$.

The *quadrea* A of the triple $\{A_1, A_2, A_3\}$ is the number

$$A = (Q_1 + Q_2 + Q_3)^2 - 2(Q_1^2 + Q_2^2 + Q_3^2).$$

The quadrea A is equal to 16 times the square of the area of the triangle $A_1A_2A_3$, and $A = 0$ precisely when the points A_1, A_2 and A_3 are collinear.

Keeping in mind the similarity between spread and cross on the one hand and $\sin^2$ and $\cos^2$ on the other hand, the following statement do not come as a surprise. The *spread law* holds for any triangle whose quadrances are non-zero:

$$\frac{s_1}{Q_1} = \frac{s_2}{Q_2} = \frac{s_3}{Q_3}.$$

The *cross law* reads: if c_3 is the cross of the lines A_2A_3 and A_1A_3 then

$$(Q_1 + Q_2 - Q_3)^2 = 4Q_1Q_2c_3.$$

Instead of the fact that the angles of a triangle add up to $180°$, here the *triple spread formula* arises for the spreads s_1, s_2 and s_3 of the triangle:

$$(s_1 + s_2 + s_3)^2 - 2(s_1^2 + s_2^2 + s_3^2) - 4s_1s_2s_3 = 0$$

The book is written in the style of a classical textbook of analytic geometry and trigonometry with an abundance of formulas. In the first chapters one will find formulas for the concurrence of the lines, the collinearity of three points, and later on formulas for the foot of the altitude, the perpendicular bisector, the orthocenter, the circumcenter, etc. The verification of properties usually boils down to plugging in coordinates. The author advocates the use of computer algebra packages for the derivation of complicated formulas. That is a rather dull way of doing geometry. It is more fun to use elementary computations in order to verify the results for some selected examples, preferably over a finite field.

An example

In this section the properties are discussed of a triangle $A_1A_2A_3$ in the plane over the field F_{13} of the integers modulo 13, where $A_1 = [2, 8]$, $A_2 = [9, 9]$, $A_3 = [10, 0]$. The quadrance $Q_1 = Q(A_2, A_3)$ of the side opposite A_1 is 4. Similarly $Q_2 = Q(A_1, A_3) = 11$ and $Q_3 = Q(A_1, A_2) = 11$, so no side is a null line and the triangle is isosceles. For the line l_1 of the side A_2A_3 opposite A_1 one gets: $l_1 = \langle 9 : 1 : 1\rangle$ and similarly $l_2 = \langle 1 : 1 : 3\rangle$ and $l_3 = \langle 12 : 7 : 11\rangle$. It follows that $s_1 = s(l_2, l_3) = 10$, $s_2 = s(l_1, l_3) = 8$ and $s_3 = s(l_1, l_2) = 8$, confirming the spread law.

The altitude h_3 on A_1A_2 is of the form $h_3 = \langle 7 : 1 : c\rangle$. As A_3 lies on h_3 one has $h_3 = \langle 7 : 1 : 8\rangle$ and its foot is the point $[3, 10]$. This can be used to compute the values of s_1 and s_2 in an alternative way. The perpendicular bisector m_3 of A_1A_2 is $m_3 = \langle 7 : 1 : 5\rangle$. In a similar way one finds the equations $h_2 = \langle 12 : 1 : 0\rangle$ and $m_2 = \langle 12 : 1 : 2\rangle$ for the altitude and the perpendicular bisector on A_1A_3, respectively. From this one can deduce that the orthocenter H is $[12, 12]$ and the center M of the circumcircle of triangle $A_1A_2A_3$ is $[11, 9]$. The centroid Z of the triangle, that is, the point of concurrence of the medians, is $[7, 10]$. One may verify that $Z = \frac{2}{3}M + \frac{1}{3}H$, so these three points lie on the same line, the Euler line. As the triangle is isosceles the Euler line coincides with the altitude h_1 on $[A_2A_3]$. For each i, the quadrance $Q(M, A_i) = 4$ and this number is denoted by K, the *quadrance K of the circumcircle*. The spread law may be extended to $\frac{s_i}{Q_i} = \frac{1}{4K}$ for each i.

One may also consider the given triangle $A_1A_2A_3$ with $A_1 = [2,8]$, $A_2 = [9,9]$, $A_3 = [10,0]$ in the plane over the field of rational numbers. In this case $M = [\frac{49}{8}, \frac{33}{8}]$, $H = [\frac{35}{4}, \frac{35}{4}]$ and $Z = [7, \frac{17}{3}]$. It can be expected that all the occurring numbers are rational. For this reason this branch of geometry is named rational trigonometry. As it may be applied to finite geometries as well, it is also called universal.

What else is new?

The book is loaded with results of classical geometry that have a counterpart in the new geometries. There is a section on proportions discussing among others the theorems of Ceva and Menelaus. Another section deals with the incircle and the excircles of a triangle.

A section that I particularly enjoyed is the one on regular stars and polygons. A regular n-star is a cyclically ordered set consisting of n lines $l_0, l_1, \ldots, l_{n-1}, l_n = l_0$ such that l_{i+1} is the reflection of l_{i-1} in l_i, for all i. It is clear that the existence of regular n-stars and regular n-gons are intimately related. One of the results is that a regular star of order 3 exists precisely when the number 3 is a nonzero square. It follows for example that there are no regular stars of order 3 in the planes over any of the fields $\mathbf{F}_3$, $\mathbf{F}_5$, $\mathbf{F}_7$, but that such stars do exist in the plane over the field $\mathbf{F}_{11}$. There is no regular star of order 3 in the plane over the rational numbers, in contrast with the situation in the plane over the real numbers.

There is a section devoted to conics that can be defined metrically, including circles and ribbons, parabolas, quadrolas and grammolas. Another section deals with cyclical quadrilaterals. Showing that so many ideas of classical geometry carry over to finite geometries is a remarkable achievement, indeed. The book certainly will kindle the reader's curiosity and lead to the invention of new results.

Editorially speaking, the book is well produced. The numbering of the items is rather unconventional. The theorems are numbered consecutively, the exercises are numbered per chapter, while definitions are not numbered at all. This does not lead to any problems as the author gives a page-reference for all citations.

I have some reservations about the dramatic implications for mathematics teaching that are claimed by the author. First of all the trigonometric functions play an important role in mathematics. Just think of the polar representation of complex numbers, the group of rotations and differential equations. The traditional introduction of the trigonometric functions as proportions in a rectangular triangle points directly to important applications. I think that the author overemphasizes the rational character of quadrance and spread. A student will hardly realize that the trigonometric functions are transcendental, let alone ever grasp the meaning of transcendence. Math instruction is about numbers and shapes. Gradually it changes into elementary algebra and geometry. One of the important goals of geometry instruction is the development of the ability to perceive spatial structure. The picture I made of the example in the previous section with all the lines drawn in, looked like a pointillistic miniature, rather than an appealing illustration of some geometric object.

In advocating his wonderful invention of the new geometry the author suppresses the beneficial employment of vectors in geometry. This statement may be clarified by the following observations. As was stated above, the non-null lines $l_1 = \langle a_1 : b_1 : c_1 \rangle$ and $l_2 = \langle a_2 : b_2 : c_2 \rangle$ are perpendicular if and only their cross c equals zero. From the formula of the spread and the equality $s = 1 - c$, for the cross we have the formula

$$c(l_1, l_2) = \frac{(a_1a_2 + b_1b_2)^2}{(a_1^2 + b_1^2)(a_2^2 + b_2^2)}.$$

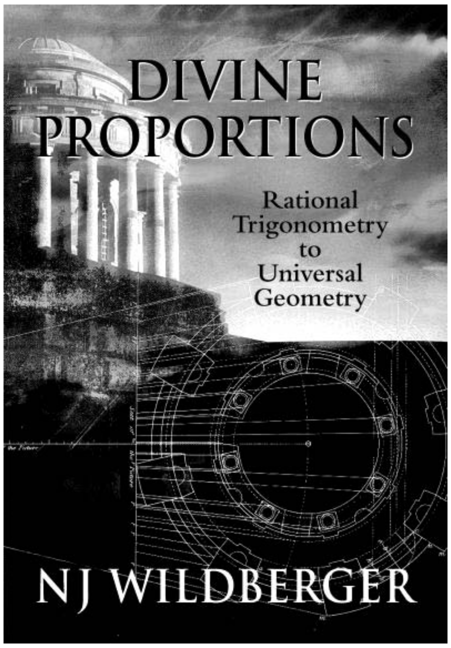
So it is quite natural to interpret (a_1, b_1) and (a_2, b_2) as normal vectors of the lines l_1 and l_2 and $a_1a_2 + b_1b_2$ as their inner product. Moreover, the vectors (a_1, a_2) and (b_1, b_2) are orthogonal (and so are the lines l_1 and l_2) as their inner product is 0. I found that the systematic use of the vector notation substantially simplifies many of the proofs.

In the final part, the applications, the plane geometry is employed to problems in solid geometry, some of which may be labeled 'applied'. I enjoyed the chapter on Platonic solids. Here the spreads between the faces of the solids are computed, resulting in the values $\frac{8}{9}, 1, \frac{8}{9}, \frac{4}{9}$ and $\frac{4}{5}$ for tetrahedron, cube, octahedron, icosahedron and dodecahedron, respectively.

In the preface the author states that the message of the book is controversial. On the Internet one may find many reviews of the book and there is a lively discussion about the merits of the new geometry. Summarizing my opinion of the book, my advise to the reader is to neglect the vast amount of formulas and to just enjoy the original approach to a new geometry. It is fun.

Acknowledgement

I thank Reinie Erné for carefully reading the manuscript and suggesting some linguistic improvements. Thanks also go to Swier Garst who converted my handmade picture of Figure 1 into an electronic picture.



N.J. Wildberger, *Divine Proportions*, 20 September 2005, Wild Egg Books, \$89,95.